GLOBAL EXISTENCE OF SOLUTIONS FOR GIERER-MEINHARDT SYSTEM WITH THREE EQUATIONS

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Abstract. This paper deals with an Gierer-Meinhardt model, with three substances, formed Reaction-Diffusion system with fractional reaction. To prove global existence for solutions of this system presents difficulties at the boundednees of fractionar term. The purpose of this paper is to prove the existence of a global solution using a boundary functionel. Our technique is based on the construction of Lyapunov functionel.

1. Introduction

In recent years, systems of Reaction-Diffusion equations have received a great deal of attention, motivated by their widespread occurrence in models of chemical and biological phenomena. These systems are divided into celebrated classes; systems with dissipation of mass and systems of Gierer-Meinhardt. In this paper we deal with this last.

In the study of the various topics from plant developmental; Meinhardt, Koch and Bernasconi [10] proposed Activator-Inhibitor models (an example is given in section 5) to describe a theory of biological pattern formation in plants (*Phyllotaxis*).

We assume a Reaction-diffusion system with three components:

$$\begin{cases} \frac{\partial u}{\partial t} - a_1 \Delta u = f(u,v,w) = \sigma - b_1 u + \frac{u^{p_1}}{v^{q_1}(w^{r_1}+c)} \\ \frac{\partial v}{\partial t} - a_2 \Delta v = g(u,v,w) = -b_2 v + \frac{u^{p_2}}{v^{q_2} w^{r_2}} \\ \frac{\partial w}{\partial t} - a_3 \Delta w = h(u,v,w) = -b_3 w + \frac{u^{p_3}}{v^{q_3} w^{r_3}} \end{cases} \quad x \in \Omega, t > 0, \tag{1.1}$$

with Neummann boundary conditions

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = \frac{\partial w}{\partial \eta} = 0 \qquad \text{on } \partial\Omega \times \{t > 0\}, \tag{1.2}$$

and the initial data

$$\begin{cases} u(0,x) = \varphi_1(x) > 0 \\ v(0,x) = \varphi_2(x) > 0 \\ w(0,x) = \varphi_3(x) > 0 \end{cases} \quad \text{on } \Omega, \tag{1.3}$$

and $\varphi_i \in C\left(\overline{\Omega}\right)$ for all $i = 1, 2, 3$.

Here Ω is an open bounded domain of class $\mathbb{C}^1$ in $\mathbb{R}^N$, with boundary $\partial\Omega$ and $\dfrac{\partial}{\partial \eta}$ denotes the outward normal derivative on $\partial\Omega$.

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$c, , p_i,\ q_i,\ r_i$: are non negative with $\sigma, b_i,\ a_i > 0$, indexes for all $i = 1, 2, 3$.

$$0 < p_1 - 1 < \max\left\{ p_2 \min\left(\frac{q_1}{q_2+1}, \frac{r_1}{r_2}, 1\right),\quad p_3 \min\left(\frac{r_1}{r_3+1}, \frac{q_1}{q_3}, 1\right)\right\}. \tag{1.4}$$

Put $A_{ij} = \frac{a_i + a_j}{2\sqrt{a_i a_j}}$ for all $i, j = 1, 2, 3$. Let α, β and γ be positive constants such that where

$$\alpha > 2\max\left\{1, \frac{b_2+b_3}{b_1}\right\}, \tag{1.5}$$

$$\frac{1}{\beta} > 2A_{12}^2, \tag{1.6}$$

and

$$\left(\frac{1}{2\beta} - A_{12}^2\right)\left(\frac{1}{2\gamma} - A_{13}^2\right) > \left(\frac{\alpha-1}{\alpha}A_{23} - A_{12}A_{13}\right)^2. \tag{1.7}$$

The main result of the paper reads as follows:

Theorem 1. *Suppose that the functions f, g and h are satisfing condition (1.4). Let $(u(t,.), v(t,.), w(t,.))$ be a solution of (1.1)-(1.3) and let:*

$$L(t) = \int_\Omega \frac{u^\alpha(t,x)}{v^\beta(t,x)\, w^\gamma(t,x)} dx. \tag{1.8}$$

Then the functional L is uniformly bounded on the interval $[0, T^*], T^* < T_{\max}$. Where $T_{\max}\left(\|u_0\|_\infty, \|v_0\|_\infty, \|w_0\|_\infty\right)$ denotes the eventual blow-up time.

Corollary 1. *Under the assumptions of theorem 1 all solutions of (1.1)-(1.3) with positive initial data in $C\left(\overline{\Omega}\right)$ are global. If in addition $b_1,\ b_2,\ b_3,\ \sigma > 0$, then (u, v, w) are uniformly bounded in $\overline{\Omega} \times [0, \infty)$.*

2. **Previous Results**

The usual norms in spaces $L^p(\Omega)$, $L^\infty(\Omega)$ and $C(\overline{\Omega})$ are denoted respectively by:

$$\|u\|_p^p = \frac{1}{|\Omega|}\int_\Omega |u(x)|^p\, dx; \tag{2.1a}$$

$$\|u\|_\infty = \max_{x\in\Omega} |u(x)|, \tag{2.1b}$$

$$\|u\|_{\mathbb{C}(\overline{\Omega})} = \max_{x\in\overline{\Omega}} |u(x)|. \tag{2.1c}$$

In 1972, following an ingenious idea of Turing.A [17], Gierer.A and Meinhardt. H [10] proposed a mathematical model for pattern formations of spatial tissue structures of hydra in morphogenesis, a biological phenomenon discovered by A.Trembley in 1744 [16].It is a system of reaction-diffusion equations of the form:

$$\begin{cases} \frac{\partial u}{\partial t} - a_1\Delta u = \sigma - \mu u + \frac{u^p}{v^q} \\ \frac{\partial v}{\partial t} - a_2\Delta v = -\nu v + \frac{u^r}{v^s} \end{cases} \quad \text{for all } x \in \Omega\ , t > 0 \tag{2.2}$$

with Neummann boundary conditions

$$\frac{\partial u}{\partial \eta} = 0, \text{ and } \frac{\partial v}{\partial \eta} = 0, \qquad x \in \partial\Omega, t > 0, \tag{2.3}$$

and initial conditions

$$\left\{ \begin{array}{l} u(x,0) = \varphi_1(x) > 0 \\ v(x,0) = \varphi_2(x) > 0 \end{array} \right. , \qquad x \in \Omega, \tag{2.4}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $a_1, a_2 > 0, \mu, \nu, \sigma > 0, p,\ q,\ r$ and s are non negative indexes with $p > 1$.

Global existence of solutions in $(0, \infty)$ is proved by Rothe in 1984 [14] with special cases: $p = 2,\ q = 1,\ r = 2,\ s = 0$ and $N = 3$.The Rothe's method cannot be applied (at least directly) to the general p, q, r, s. It is desirable to consider the p, q, r, s originally proposed by Gierer-Meinhardt. Wu and Li [8] obtained the same results for the problem (2.1)-(2.3) so long as u, v^{-1} and σ are suitably small. Mingde, Shaohua and Yuchun [11] show that the solutions of this problem are bounded all the time for each pair of initial values if

$$\frac{p-1}{r} < \frac{q}{s+1}, \tag{2.5a}$$

$$\frac{p-1}{r} < 1. \tag{2.5b}$$

Masuda. K and Takahashi. K [9] we consider a more general system for (u, v) :

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t} - a_1 \Delta u = \sigma_1(x) - \mu u + \rho_1(x,u) \frac{u^p}{v^q} \\ \frac{\partial v}{\partial t} - a_2 \Delta v = \sigma_2(x) - \nu v + \rho_2(x,u) \frac{u^r}{v^s} \end{array} \right. \tag{2.6}$$

with $\sigma_1, \sigma_2 \in C^1\left(\overline{\Omega}\right),\ \sigma_1 \geq 0, \sigma_2 \geq 0, \rho_1, \rho_2 \in C^1\left(\overline{\Omega} \times \overline{\mathbb{R}}_+^2\right) \cap L^\infty\left(\overline{\Omega} \times \overline{\mathbb{R}}_+^2\right)$ satisfying $\rho_1 \geq 0, \rho_2 > 0$ and $p,\ q,\ r$,s are nonnegative constants satisfying (2.5a). Obviously, (2.4) system is a special case of (2.6) system. In 1987, Masuda. K and Takahashi. K [9] extended the result to $\frac{p-1}{r} < \frac{2}{N+2}$ under the condition $\sigma_1 > 0$.In 2006 Jiang.H [7] under the conditions (2.5a) - (2.5b) , $\varphi_1, \varphi_2 \in W^{2,l}(\Omega), l > \max\{N, 2\}$, $\frac{\partial \varphi_1}{\partial \eta} = \frac{\partial \varphi_2}{\partial \eta} = 0$ on $\partial\Omega$ and $\varphi_1 \geq 0, \varphi_2 > 0$ in $\overline{\Omega}$.Then (2.6) system has a unique nonnegative global solution (u, v) satisfying (2.3)-(2.4).

3. **Preliminary Observations**

It is well-known that to prove global existence of solutions to $(1.1) - (1.3)$ (see Henry [6]), it suffices to derive a uniform estimate of $\|f(u,v,w)\|_p$, $\|g(u,v,w)\|_p$ and $\|h(u,v,w)\|_p$on $[0; T_{\max}[$ in the space $L^p(\Omega)$ for some $p > N/2$.Our aim is to construct polynomial Lyapunov functionals allowing us to obtain L^p- bounds on $u; v$ and w that lead to global existence. Since the functions f, g and h are continuously differentiable on $\mathbb{R}_+^3$, then for any initial data in $C(\overline{\Omega})$, it is easy to check directly their Lipschitz continuity on bounded subsets of the domain of a fractional power of the operator

$$O = -\begin{pmatrix} a_1\Delta & 0 & 0 \\ 0 & a_2\Delta & 0 \\ 0 & 0 & a_3\Delta \end{pmatrix}. \tag{3.1}$$

Under these assumptions, the following local existence result is well known (see Friedman [3] and Pazy [13]).

Proposition 1. *The system* $(1.1)-(1.3)$ *admits a unique, classical solution* $(u; v; w)$ *on* $(0, T_{\max}[\times\Omega$.

$$\text{If } T_{\max} < \infty \text{ then } \lim_{t \nearrow T_{\max}} \left(\|u(t,.)\|_\infty + \|v(t,.)\|_\infty + \|w(t,.)\|_\infty\right) = \infty. \tag{3.2}$$

Remark 1. *This proposition seems to be well-known (Dan Henry* [6]*). Neverthless we could not find it in the literature in the form stated here and stated in the book of Franz Rothe (*[14] *pp: 111-118 with proof). Usually the explosion property* (3.2) *is only stated for some norm involving smoothness, but not the* $L_\infty-$*norm.*

4. **Proofs**

For the proof of theorem 1, we need a preparatory Lemmas, which are proved in the appendix.

Lemma 1. *Assume that* p, q, r, s, m, *and* n *satisfying*

$$\frac{p-1}{r} < \min\left(\frac{q}{s+1}, \frac{m}{n}, 1\right).$$

For all h, l, α, β, $\gamma > 0$, *there exist* $C = C\left(h, l, \alpha, \beta, \gamma\right) > 0$ *and* $\theta = \theta\left(\alpha\right) \in (0,1)$, *such that*

$$\alpha \frac{x^{p-1+\alpha}}{y^{q+\beta} z^{m+\gamma}} \leq \beta \frac{x^{r+\alpha}}{y^{s+1+\beta} z^{n+\gamma}} + C\left(\frac{x^{\alpha}}{y^{\beta} z^{\gamma}}\right)^{\theta}, \qquad x \geq 0, y \geq h, z \geq l \tag{4.1}$$

Lemma 2. *Let* $\mu, T > 0$ *and* $f_j = f_j(t)$ *be a non-negative integrable function on* $[0,T)$*and* $0 < \theta_j < 1$ $\;(j = 1, ..., J)$*. Let* $W = W(t)$*be a positive function on* $[0,T)$ *satisfying the differential inequality*

$$\frac{dW\left(t\right)}{dt} \leq -\mu W\left(t\right) + \sum_{j=1}^{J} f_j(t) W^{\theta_j}\left(t\right), \qquad 0 \leq t < T. \tag{4.2}$$

Then, we obtain that

$$W\left(t\right) \leq \kappa, \qquad 0 \leq t < T, \tag{4.3}$$

where κ *is the maximal root of the following algebraic equation:*

$$x - \sum_{j=1}^{J}\left(\sup_{0<t<T}\int_0^t e^{-\mu(t-\xi)} f_j(\xi) d\xi\right) x^{\theta_j} = W\left(0\right). \tag{4.4}$$

Moreover, if $T = +\infty$, *then*

$$\limsup_{t\nearrow\infty} W\left(t\right) \leq \kappa_\infty,$$

where κ_∞ *is the maximal root of the following algebraic equation:*

$$x - \sum_{j=1}^{J}\left(\limsup_{t\nearrow\infty}\int_0^t e^{-\mu(t-\xi)} f_j(\xi) d\xi\right) x^{\theta_j} = 0.$$

Lemma 3. *Let* $\left(u\left(t,.\right), v\left(t,.\right), w\left(t,.\right)\right)$ *be a solution of (1.1)-(1.3), then for any* (t,x) *in* $(0, T_{\max}[\times\Omega$ *we get*

$$\left\{\begin{array}{l} u(t,x) \geq e^{-b_1 t}\min(\varphi_1(x)) > 0, \\ v(t,x) \geq e^{-b_2 t}\min(\varphi_2(x)) > 0, \\ w(t,x) \geq e^{-b_3 t}\min(\varphi_3(x)) > 0. \end{array}\right. \tag{4.5}$$

proof of theorem 1. Differentiating $L(t)$ with respect to t yields

$$\begin{aligned} L'(t) &= \int_\Omega \frac{d}{dt}\left(\frac{u^\alpha}{v^\beta w^\gamma}\right)dx, \\ &= \int_\Omega \left(\alpha\frac{u^{\alpha-1}}{v^\beta w^\gamma}\partial_t u - \beta\frac{u^\alpha}{v^{\beta+1}w^\gamma}\partial_t v - \gamma\frac{u^\alpha}{v^\beta w^{\gamma+1}}\partial_t w\right)dx, \end{aligned}$$

replacing $\partial_t u$, $\partial_t v$ and $\partial_t w$ with its values in (1.1), we get

$$\begin{aligned} L'(t) &= \int_\Omega \begin{pmatrix} a_1\alpha\frac{u^{\alpha-1}}{v^\beta w^\gamma}\Delta u - a_2\beta\frac{u^\alpha}{v^{\beta+1}w^\gamma}\Delta v - a_3\gamma\frac{u^\alpha}{v^\beta w^{\gamma+1}}\Delta w \\ -b_1\alpha\frac{u^\alpha}{v^\beta w^\gamma} + b_2\beta\frac{u^\alpha}{v^\beta w^\gamma} + b_3\gamma\frac{u^\alpha}{v^\beta w^\gamma} \\ +\alpha\frac{u^{p_1+\alpha-1}}{v^{q_1+\beta}w^\gamma(w^{r_1}+c)} - \beta\frac{u^{p_2+\alpha}}{v^{q_2+\beta+1}w^{r_2+\gamma}} - \gamma\frac{u^{p_3+\alpha}}{v^{q_3+\beta}w^{r_3+\gamma+1}} + \sigma\alpha\frac{u^{\alpha-1}}{v^\beta w^\gamma} \end{pmatrix}dx, \\ &= I + J, \end{aligned}$$

where I contains laplacian terms and J contains the other terms

$$\begin{aligned} I &= a_1\alpha\int_\Omega \frac{u^{\alpha-1}}{v^\beta w^\gamma}\Delta u dx - a_2\beta\int_\Omega\frac{u^\alpha}{v^{\beta+1}w^\gamma}\Delta v dx - a_3\gamma\int_\Omega\frac{u^\alpha}{v^\beta w^{\gamma+1}}\Delta w dx, \\ J &= (-b_1\alpha + b_2\beta + b_3\gamma)L(t) \\ &\quad +\alpha\int_\Omega \frac{u^{p_1+\alpha-1}}{v^{q_1+\beta}w_3^\gamma(w^{r_1}+c)}dx - \beta\int_\Omega\frac{u^{p_2+\alpha}}{v^{q_2+\beta+1}w^{r_2+\gamma}}dx - \gamma\int_\Omega\frac{u^{p_3+\alpha}}{v^{q_3+\beta}w^{r_3+\gamma+1}}dx \\ &\quad +\sigma\alpha\int_\Omega\frac{u^{\alpha-1}}{v^\beta w^\gamma}dx. \end{aligned}$$

Starting with estimation of I:

Using Green's formula for terms $\int_\Omega \frac{u^{\alpha-1}}{v^\beta w^\gamma}\Delta u dx$, $\int_\Omega \frac{u^\alpha}{v^{\beta+1}w^\gamma}\Delta v dx$ and $\int_\Omega \frac{u^\alpha}{v^\beta w^{\gamma+1}}\Delta w dx$ we get

$$\begin{aligned} I &= \int_\Omega \begin{pmatrix} -a_1\alpha(\alpha-1)\frac{u^{\alpha-2}}{v^\beta w^\gamma}|\nabla u|^2 + a_1\alpha\beta\frac{u^{\alpha-1}}{v^{\beta+1}w^\gamma}\nabla u\nabla v + a_1\alpha\gamma\frac{u^{\alpha-1}}{v^\beta w^{\gamma+1}}\nabla u\nabla w \\ +a_2\beta\alpha\frac{u^{\alpha-1}}{v^{\beta+1}w^\gamma}\nabla u\nabla v - a_2\beta(\beta+1)\frac{u^\alpha}{v^{\beta+2}w^\gamma}|\nabla v|^2 - a_2\beta\gamma\frac{u^\alpha}{v^{\beta+1}w^{\gamma+1}}\nabla v\nabla w \\ +a_3\gamma\alpha\frac{u^{\alpha-1}}{v^\beta w^{\gamma+1}}\nabla u\nabla w - a_3\gamma\beta\frac{u^\alpha}{v^{\beta+1}w^{\gamma+1}}\nabla v\nabla w - a_3\gamma(\gamma+1)\frac{u^\alpha}{v^\beta w^{\gamma+2}}|\nabla w|^2 \end{pmatrix}dx, \\ &= -\int_\Omega\left[\frac{u^{\alpha-2}}{v^{\beta+2}w^{\gamma+2}}(QT)\cdot T\right]dx, \end{aligned}$$

where

$$Q = \begin{pmatrix} a_1\alpha(\alpha-1) & -\alpha\beta\frac{a_1+a_2}{2} & -\alpha\gamma\frac{a_1+a_3}{2} \\ -\alpha\beta\frac{a_1+a_2}{2} & a_2\beta(\beta+1) & \beta\gamma\frac{a_2+a_3}{2} \\ -\alpha\gamma\frac{a_1+a_3}{2} & \beta\gamma\frac{a_2+a_3}{2} & a_3\gamma(\gamma+1) \end{pmatrix}.$$

Q is matrix of quadratic form compared to: $vw\nabla u$, $uw\nabla v$ and $uv\nabla w$, which is written in the matric form:$T = (vw\nabla u, uw\nabla v, uv\nabla w)^t$.
Q is definite positive if, and only if all its principal successive determinants are positive. To see this, we have:
1. $\Delta_1 = a_1\alpha(\alpha-1) > 0$. Using (1.5), we get $\Delta_1 > 0$.
2. $\Delta_2 = \begin{vmatrix} a_1\alpha(\alpha-1) & -\alpha\beta\frac{a_1+a_2}{2} \\ -\alpha\beta\frac{a_1+a_2}{2} & a_2\beta(\beta+1)\end{vmatrix} = \alpha^2\beta^2 a_1a_2\left(\frac{\alpha-1}{\alpha}\frac{\beta+1}{\beta} - A_{12}^2\right)$. Using (1.5) and (1.6), we get $\Delta_2 > 0$.
3. Using theorem 1 in S.abdelmalek and S. Kouachi [1] we get $(\alpha-1)\Delta_3 = (\alpha-1)|Q| = \alpha(\alpha\gamma\beta)^2 a_1a_2a_3(\left(\frac{\alpha-1}{\alpha}\frac{\beta+1}{\beta} - A_{12}^2\right)\left(\frac{\alpha-1}{\alpha}\frac{\gamma+1}{\gamma} - A_{13}^2\right) - \left(\frac{\alpha-1}{\alpha}A_{23} - A_{12}A_{13}\right)^2)$.
Using (1.5)-(1.7), we get $\Delta_3 > 0$.

Consequently we have $I \leq 0, \qquad \forall\ (t,x)\ \in\ [0,T^*] \times \Omega.$
Now we estimate J :

$$\begin{aligned} J &= (-b_1\alpha + b_2\beta + b_3\gamma)\, L\,(t) \\ &+\alpha \int_\Omega \frac{u^{p_1+\alpha-1}}{v^{q_1+\beta} w^\gamma (w^{r_1}+c)} dx - \beta \int_\Omega \frac{u^{p_2+\alpha}}{v^{q_2+\beta+1} w^{r_2+\gamma}} dx - \gamma \int_\Omega \frac{u^{p_3+\alpha}}{v^{q_3+\beta} w^{r_3+\gamma+1}} dx \\ &+\sigma\alpha \int_\Omega \frac{u^{\alpha-1}}{v^\beta w^\gamma} dx. \end{aligned}$$

According to the maximum principle, there exists C_0 dependant on φ_1, φ_2 and φ_3 such that $v, w \geq C_0 > 0$, then we have

$$\frac{u^{\alpha-1}}{v^\beta w^\gamma} = \left(\frac{u^\alpha}{v^\beta w^\gamma}\right)^{\frac{\alpha-1}{\alpha}} \left(\frac{1}{v}\right)^{\frac{\beta}{\alpha}} \left(\frac{1}{w}\right)^{\frac{\gamma}{\alpha}} \leq \left(\frac{u^\alpha}{v^\beta w^\gamma}\right)^{\frac{\alpha-1}{\alpha}} \left(\frac{1}{C_0}\right)^{\frac{\beta+\gamma}{\alpha}},$$

then

$$\frac{u^{\alpha-1}}{v^\beta w^\gamma} \leq C_2 \left(\frac{u^\alpha}{v^\beta w^\gamma}\right)^{\frac{\alpha-1}{\alpha}} \qquad \text{where } C_2 = \left(\frac{1}{C_0}\right)^{\frac{\beta+\gamma}{\alpha}},$$

we have

$$\begin{aligned} J &= (-b_1\alpha + b_2\beta + b_3\gamma)\, L\,(t) \\ &+\alpha \int_\Omega \frac{u^{p_1+\alpha-1}}{v^{q_1+\beta} w^\gamma\, (w^{r_1}+c)} dx - \beta \int_\Omega \frac{u^{p_2+\alpha}}{v^{q_2+\beta+1} w^{r_2+\gamma}} dx - \gamma \int_\Omega \frac{u^{p_3+\alpha}}{v^{q_3+\beta} w^{r_3+\gamma+1}} dx \\ &+\sigma\alpha \int_\Omega \frac{u^{\alpha-1}}{v^\beta w^\gamma} dx. \end{aligned}$$

Using lemma 1, $\forall (t,x) \in [0,T^*]\,[\times\Omega$ we get

$$\alpha \frac{u^{p_1+\alpha-1}}{v^{q_1+\beta} w^\gamma (w^{r_1}+c)} \leq \alpha \frac{u^{p_1+\alpha-1}}{v^{q_1+\beta} w^{\gamma+r_1}} \leq \beta \frac{u^{p_2+\alpha}}{v^{q_2+\beta+1} w^{r_2+\gamma}} + C \left(\frac{u^\alpha}{v^\beta w^\gamma}\right)^\theta \tag{4.6}$$

or

$$\alpha \frac{u^{p_1+\alpha-1}}{v^{q_1+\beta} w^{\gamma+r_1}} \leq \gamma \frac{u^{p_3+\alpha}}{w^{r_3+1+\gamma} v^{q_3+\beta}} + C \left(\frac{u^\alpha}{v^\beta w^\gamma}\right)^\theta \tag{4.7}$$

Using (4.6) or (4.7) then

$$J \leq (-b_1\alpha + b_2\beta + b_3\gamma)\, L\,(t) + \int_\Omega C \left(\frac{u^\alpha}{v^\beta w^\gamma}\right)^\theta dx + \alpha\sigma \int_\Omega C_2 \left(\frac{u^\alpha}{v^\beta w^\gamma}\right)^{\frac{\alpha-1}{\alpha}} dx.$$

Applying Hölder's inequality, for all t in $[0,T^*]$ we obtain

$$\int_\Omega C \left(\frac{u^\alpha}{v^\beta w^\gamma}\right)^\theta dx \leq \left(\int_\Omega \left(\frac{u^\alpha}{v^\beta w^\gamma}\right) dx\right)^\theta \left(\int_\Omega C^{\frac{1}{1-\theta}} dx\right)^{1-\theta},$$

then

$$\int_\Omega C \left(\frac{u^\alpha}{v^\beta w^\gamma}\right)^\theta dx \leq C_3 L^\theta(t), \qquad \text{where } C_3 = C\,|\Omega|^{1-\theta}.$$

We have

$$\int_\Omega C_2 \left(\frac{u^\alpha}{v^\beta w^\gamma}\right)^{\frac{\alpha-1}{\alpha}} dx \leq \left(\int_\Omega \left(\frac{u^\alpha}{v^\beta w^\gamma}\right) dx\right)^{\frac{\alpha-1}{\alpha}} \left(\int_\Omega (C_2)^\alpha\, dx\right)^{\frac{1}{\alpha}},$$

then

$$\int_\Omega C_2\left(\frac{u^\alpha}{v^\beta w^\gamma}\right)^{\frac{\alpha-1}{\alpha}}dx \le C_4 L^{\frac{\alpha-1}{\alpha}}(t) \quad \text{where } C_4 = C_2\left|\Omega\right|^{\frac{1}{\alpha}},$$

we get

$$J \le \left(-b_1\alpha + b_2\beta + b_3\gamma\right)L(t) + C_3 L^\theta(t) + \alpha\sigma C_4 L^{\frac{\alpha-1}{\alpha}}(t),$$

which implies

$$J \le \left(-b_1\alpha + b_2\beta + b_3\gamma\right)L(t) + C_5\left(L^\theta(t) + \alpha\sigma L^{\frac{\alpha-1}{\alpha}}(t)\right).$$

Thus under conditions (1.5), (1.6) and (1.7), we obtain

$$L'(t) \le \left(-b_1\alpha + b_2\beta + b_3\gamma\right)L(t) + C_5\left(L^\theta(t) + \alpha\sigma L^{\frac{\alpha-1}{\alpha}}(t)\right),$$

since $-b_1\alpha + b_2\beta + b_3\gamma < 0$ and Using lemma 2 we deduce that $L(t)$ is bounded on $(0, T_{\max}[$ ie $L(t) \le \kappa$, where κ dependent on φ_1, φ_2 and φ_3. □

Proof of Corollary 1. Since $L(t)$ is bounded on $(0, T_{\max}[$ and the functions $\frac{u^{p_1}}{v^{q_1}(w^{r_1}+c)}$ $\frac{u^{p_2}}{v^{q_2}w^{r_2}}$ and $\frac{u^{p_3}}{v^{q_3}w^{r_3}}$ are in $L^\infty((0,\ T_{\max}),\ L^m(\Omega))$ for all $m > \frac{N}{2}$, then as a consequence of the arguments in Henry. D [6] or Haraux. A and Kirane. M [5] we conclude the solution of the system (1-1)-(1-7) is global and uniformly bounded on $\Omega \times (0, +\infty)$. □

5. Example

In this section we will examine a particular activator-inhibitor model in order to illusrate the applicability of corollary 1 and proposition 1.We assume that all reactions take place in a bounded domain Ω with a smooth boundary $\partial\Omega$.

Example 1. *The model proposed by Meinhardt, Koch and Bernasconi* [10] *to describe a theory of biological pattern formation in plants (Phyllotaxis), where u, v and w are the concentrations of three substances; called activator (u) and inhibitors (v and w) is:*

$$\begin{cases} \frac{\partial u}{\partial t} - a_1\frac{\partial^2 u}{\partial x^2} = -b_1 u + \frac{a^2}{v(w+\kappa_u)} + \sigma, \\ \frac{\partial v}{\partial t} - a_2\frac{\partial^2 v}{\partial x^2} = -b_2 v + u^2, \\ \frac{\partial w}{\partial t} - a_3\frac{\partial^2 w}{\partial x^2} = -b_3 w + u, \end{cases} \quad \textit{for all } x \in \Omega,\ t > 0. \tag{5.1}$$

Proposition 2. *Solutions of* (5.1) *with boundary conditions* (1.2) *and nonnegative uniformly bounded initial data* (1.3) *exist globally.*

Proof. This model is a special case of our general model (1.1), where $p_1 = 2, q_1 = 1, r_1 = 1, p_2 = 2, q_2 = 0, r_2 = 0, p_3 = 1, q_3 = 0, r_3 = 0$. These indexes realize the conditions of global existence: $\frac{p_1-1}{p_2} < \min\left(\frac{q_1}{q_2+1}, \frac{r_1}{r_2}, 1\right)$. □

Remark 2. *The system described by equations* (5.1) *exhibits all the essential features of phyllotaxis.*

6. Appendix

The purpose of this appendix is to prove lemma 1, lemma 2 and lemma 3 in section 4 which we have used in the proof of theorem 1.

Proof of Lemma 1. For all $x \geq 0, y \geq h, z \geq l$ we have from the inequality (4.1)

$$\alpha \frac{x^{p-1}}{y^q z^m} \leq \beta \frac{x^r}{y^{s+1} z^n} + C \left(\frac{x^\alpha}{y^\beta z^\gamma} \right)^{\theta - 1} \tag{6.1}$$

and we can write

$$\alpha \frac{x^{p-1}}{y^q z^m} = \alpha \beta^{-\frac{p-1}{r}} \left(\beta \frac{x^r}{y^{s+1} z^n} \right)^{\frac{p-1}{r}} y^{\frac{(s+1)(p-1)}{r} - q} z^{\frac{n(p-1)}{r} - m} .$$

For each ϵ realize: $0 < \epsilon < \min \left(\frac{q}{s+1}, \frac{m}{n}, 1 \right) - \frac{p-1}{r}$

$$\alpha \frac{x^{p-1}}{y^q z^m} = \alpha \beta^{-\frac{p-1}{r}} \left(\beta \frac{x^r}{y^{s+1} z^n} \right)^{\frac{p-1}{r} + \epsilon} \left(\beta \frac{x^r}{y^{s+1} z^n} \right)^{-\epsilon} v^{\frac{(s+1)(p-1)}{r} - q} z^{\frac{n(p-1)}{r} - m}.$$

Then also

$$\begin{aligned}
\alpha \frac{x^{p-1}}{y^q z^m} &= \alpha (\beta)^{-\frac{p-1}{r} - \epsilon} \left(\beta \frac{x^r}{y^{s+1} z^n} \right)^{\frac{p-1}{r} + \epsilon} \left(\frac{1}{x^\alpha} \right)^{\frac{r\epsilon}{\alpha}} (y)^{\frac{(s+1)(p-1)}{r} - q + \epsilon(s+1)} z^{\frac{n(p-1)}{r} - m + \epsilon n}, \\
&\leq \alpha (\beta)^{-\frac{p-1}{r} - \epsilon} \left(\beta \frac{x^r}{y^{s+1} z^n} \right)^{\frac{p-1}{r} + \epsilon} \left(\frac{1}{x^\alpha} \right)^{\frac{r\epsilon}{\alpha}} (h)^{\frac{(s+1)(p-1)}{r} - q + \epsilon(s+1)} l^{\frac{n(p-1)}{r} - m + \epsilon n}, \\
&\leq \alpha (\beta)^{-\frac{p-1}{r} - \epsilon} \left(\beta \frac{x^r}{y^{s+1} z^n} \right)^{\frac{p-1}{r} + \epsilon} \left(\frac{1}{x^\alpha} \right)^{\frac{r\epsilon}{\alpha}} (h)^{\frac{(s+1)(p-1)}{r} - q + \epsilon(s+1)} \times \\
&\quad l^{\frac{n(p-1)}{r} - m + \epsilon n} \left(\frac{y}{h} \right)^{\frac{\beta r \epsilon}{\alpha}} \left(\frac{z}{l} \right)^{\frac{\gamma r \epsilon}{\alpha}}, \\
&\leq C_1 \left(\beta \frac{x^r}{y^{s+1} z^n} \right)^{\frac{p-1}{r} + \epsilon} \left(\frac{y^\beta z^\gamma}{x^\alpha} \right)^{\frac{r\epsilon}{\alpha}}, \qquad (6.2)
\end{aligned}$$

where

$$C_1 = \alpha (\beta)^{-\frac{p-1}{r} - \epsilon} h^{\frac{(s+1)(p-1)}{r} - q + \epsilon(s+1) - \frac{\beta r \epsilon}{\alpha}} l^{\frac{(n)(p-1)}{r} - m + \epsilon n - \frac{\gamma r \epsilon}{\alpha}}.$$

Using Young's inequality for (6.2) with taking $C = C_1^{1 + \frac{p-1+r\epsilon}{r-(p-1)-r\epsilon}}$ and $\theta = 1 - \frac{r\epsilon}{\alpha\left(1 - \frac{p-1}{r} - \epsilon\right)}$ where ϵ is sufficiently small, we get inequality (6.1). □

Proof of Lemma 2. This lemma is proved in [Masuda.K and Takahashi. K [9], Lemma 2.2]. □

Proof of Lemma 3. Immediate from the maximum principle. □

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